A note on the Arens' space and sequential fan *

Shou Lin

Department of Mathematics, Ningde Teachers' College, Ningde, Fujian 352100, People's Republic of China

Received 28 November 1994; revised 31 March 1995, 28 March 1996, 7 February 1997

Abstract

In this paper we discuss the spaces containing a subspace having the Arens' space or sequential fan as its sequential coreflection. A sequential coreflection of a space which is weakly first-countable is characterized, and some generalized metric spaces which contain no Arens' space or sequential fan as its sequential coreflection are studied. © 1997 Elsevier Science B.V.

Keywords: Arens' space; Sequential fan; Sequential coreflection; k-network; Universal cs-network; Sequentially open network; Fréchet space; Sequential space; α4-space

AMS classification: 54A20; 54D55; 54F65; 54E99

1. Introduction

g-metrizable spaces and R-spaces play an important role in metrization theory. We know that every metric space is a g-metrizable space, and every g-metrizable space is an R-space. Further relationships among these spaces can be characterized by the canonical quotient spaces which are Arens’ space S2 and sequential fan S(ω). For example.

Theorem 1.1 [20]. A space is a metrizable space if and only if it is a g-metrizable space containing no (closed) copy of S2.

Theorem 1.2 [11]. A space is a g-metrizable space if and only if it is a k and R-space containing no (closed) copy of S(ω).

Using these concrete spaces S2 and S(ω) we can analyze the gaps among some generalized metric spaces. Spaces containing a copy of S2 or S(ω) and their applications

* Supported by the NNSF and the NSF of Fujian Province, People’s Republic of China.
have been studied in [11,14–17,20]. Since \( S_2 \) and \( S(\omega) \) are all sequential spaces, this encourages us to discuss the spaces containing a subspace having \( S_2 \) or \( S(\omega) \) as its sequential coreflection and those which do not. We obtain that

**Theorem 1.3.** A regular space has a \( \sigma \)-locally finite sequentially open network if and only if it has a \( \sigma \)-locally finite universal cs-network contains no (closed) subspace having \( S_2 \) as its sequential coreflection (Corollary 2.9).

**Theorem 1.4.** A regular space has a \( \sigma \)-locally finite universal cs-network if and only if it has a \( \sigma \)-locally finite cs-network and contains no (closed) subspace having \( S(\omega) \) as its sequential coreflection (Theorem 3.15).

**Theorem 1.5.** Suppose \( X \) is a quotient s-image of a metric space. \( X \) has a point-countable base if and only if \( X \) contains no (closed) copy of \( S_2 \) and \( S(\omega) \) (Corollary 3.10).

In this paper all spaces are \( T_2 \), \( \mathcal{N} \) denotes the set of all natural numbers. The Arens' space \( S_2 \) [1] and sequential fan \( S(\omega) \) [5] are defined as follows. Let \( T_0 = \{ a_n: n \in \mathcal{N} \} \) be a sequence converging to \( x \notin T_0 \) and let each \( T_n \) \((n \in \mathcal{N})\) be a sequence converging to \( a_n \notin T_n \). Let \( T \) be the topological sum of \( \{T_n \cup \{a_n\}: n \in \mathcal{N} \} \). Thus \( S_2 = \{x\} \cup (\bigcup \{T_n: n \geq 0\}) \) is a quotient space obtained from the topological sum of \( T_0 \) and \( T \) by identifying each \( a_n \in T_0 \) with \( a_n \in T \). Also, \( S(\omega) = \{x\} \cup (\bigcup \{T_n: n \in \mathcal{N} \}) \) is a quotient space obtained from \( T \) by identifying all the points \( a_n \in T \) to the point \( x \).

2. On the Arens' space \( S_2 \)

For a space \( X \) and \( x \in P \subset X \), \( P \) is a sequential barrier at \( x \) if, whenever \( \{x_n\} \) is a sequence converging to \( x \) in \( X \), then \( x_n \in P \) for all but finitely many \( n \in \mathcal{N}; \) equivalently, \( x_n \in P \) for infinitely many \( n \in \mathcal{N} \). \( P \) is sequentially open in \( X \) if \( P \) is a sequential barrier at each of its points, and is sequentially closed in \( X \) if its complement is sequential open.

A space \( X \) is called a sequential space [7] if each sequentially open subset of \( X \) is open in \( X \). Thus the topology is naturally definable using convergent sequences, and two sequential topologies on the same set \( X \) are the same if and only if they have the same convergent sequences. Each space \((X, \tau)\) has a sequential coreflection, which we denote \((X, \sigma_\tau)\) or \( \sigma X \) if there is no danger of confusion. As is well known, \( \sigma X \) is a sequential space, and \( B \in \sigma_\tau \) if and only if \( B \) is sequentially open in \( X \); also, \( X \) and \( \sigma X \) have the same convergent sequences.

**Definition 2.1.** Call a subspace of a space a comb (at a point \( x \)) if it consists of a point \( x \), a sequence \( \{x_n\} \) converging to \( x \), and disjoint sequences converging individually to each \( x_n \). Call a subset of a comb a diagonal if it is a convergent sequence meeting
infinitely many of the sequences converging to the individual \( x_n \) and converges to some point in the comb.

\( S_2 \) is comb without a diagonal.

**Lemma 2.2.** For a space \( X \), \( \sigma X \) is homeomorphic to \( S_2 \) if and only if \( X \) is a comb without a diagonal.

**Proof.** Suppose \( \sigma X \) is homeomorphic to \( S_2 \). Since \( \sigma X \) and \( X \) have the same convergent sequences and \( S_2 \) is a comb without a diagonal, \( X \) is a comb without a diagonal. Conversely, suppose \( X \) is a comb without a diagonal. Since \( \sigma X \) is sequential, \( \sigma X \) is homeomorphic to \( S_2 \). \( \square \)

A space \( X \) is called a Fréchet space \([7]\) (or a Fréchet–Urysohn space) if, whenever \( x \in \overline{\text{cl}_X(A)} \), there is a sequence in \( A \) converging to \( x \) in \( X \). Every Fréchet space is sequential, and the sequential space \( S_2 \) is not Fréchet. To characterize the Fréchet property of the sequential coreflection of a space, we introduce the following notations. For a space \( X \) and \( A \subset X \), define that

\[
\text{cl}_s(A) = \{x \in X : \text{there is a sequence in } A \text{ converging to } x \}.
\]

The following is well known and easy to show.

**Lemma 2.3.** The following are equivalent for a space \( X \):

1. \( \sigma X \) is a Fréchet space.
2. \( \text{cl}_s(A) = \text{cl}_s(A) \) for each \( A \subset X \).
3. \( \text{cl}_s(A) \) is sequentially closed in \( X \) for each \( A \subset X \).

It is easy to see from this that \( \sigma X \) is a Fréchet space if and only if every sequential barrier at any point \( x \) in \( X \) contains a sequentially open subspace containing \( x \).

**Theorem 2.4.** The following are equivalent for a space \( X \):

1. \( \sigma X \) is a Fréchet space.
2. Every comb at \( x \) of \( X \) has a diagonal converging to \( x \) for each \( x \in X \).
3. Every comb of \( X \) has a diagonal.
4. \( X \) contains no subspace having \( S_2 \) as its sequential coreflection.

**Proof.** We only need to prove that (4) \( \Rightarrow \) (1). Suppose \( \sigma X \) is not Fréchet. By Lemma 2.3, there is a subset \( A \) of \( X \) such that \( \text{cl}_s(A) \) is not closed in \( \sigma X \). Since \( \sigma X \) is sequential, there exists a sequence \( \{x_n\} \) in \( \text{cl}_s(A) \) converging to \( x \in X \setminus \text{cl}_s(A) \). We can assume that the \( x_n \)'s are all distinct and \( x_n \notin A \). Since \( X \) is \( T_2 \), let \( \{V_n\} \) be a sequence of pairwise disjoint open subsets of \( X \) with each \( x_n \in V_n \). For each \( n \in \mathcal{N} \), there is a sequence \( \{x_{nm}\} \) in \( A \cap V_n \) converging to \( x_n \) in \( X \). Put

\[
C = \{x\} \cup \{x_n : n \in \mathcal{N}\} \cup \{x_{nm} : n, m \in \mathcal{N}\}.
\]
Then $C$ is a comb at $x$ of $X$. By (4), $\sigma C$ is not homeomorphic to $S_2$. By Lemma 2.2, $C$ has a diagonal. Let $\{y_k\}$ be a diagonal of $C$ which converges to $y$ in $C$. If $y \neq x$, then $y \in V_i$ for some $i \in \mathcal{N}$, and $y_k \in V_i$ for some $j \in \mathcal{N}$ and all $k \geq j$, a contradiction. Thus $C$ has a diagonal converging to $x$, hence $x \in \text{cl}_\sigma(A)$, a contradiction. Therefore $\sigma X$ is Fréchet. $\square$

A point $x$ in a space $X$ is called regular $G_6$ if there is a sequence of neighborhoods of $x$ in $X$ such that the intersections of their closures is $\{x\}$.

**Lemma 2.5.** Let $X$ be a space in which each point is regular $G_6$. If $X$ contains no closed subspace having $S_2$ as its sequential coreflection, then $\sigma X$ is a Fréchet space.

**Proof.** By Theorem 2.4, we only need to show that if $X$ contains a subspace $S$ such that $\sigma S$ is homeomorphic to $S_2$, then $S$ contains a closed subspace $T$ of $X$ such that $\sigma T$ is homeomorphic to $S_2$. Let $S = \{x\} \cup \{x_n: n \in \mathcal{N}\} \cup \{x_{nm}: n, m \in \mathcal{N}\}$. Take a sequence $\{G_k\}$ of open neighborhoods of $x$ in $X$ such that each $G_{k+1} \subset G_k$ and $\{x\} = \bigcap \{\text{cl}(G_k): k \in \mathcal{N}\}$. Since the sequence $\{x_n\}$ converges to $x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with each $x_{n_k} \in G_k$. Since the sequence $\{x_{n_km}\}$ converges to $x_{n_k}$ for each $m \in \mathcal{N}$, there is $m_k \in \mathcal{N}$ such that $x_{n_km} \in G_k$ if $m \geq m_k$. Put

$$T = \{x\} \cup \{x_{n_k}: k \in \mathcal{N}\} \cup \{x_{n_km}: k \in \mathcal{N}, \ m \geq m_k\}.$$

If $p \in X \setminus T$, then $p \in X \setminus \text{cl}(G_k)$ for some $k \in \mathcal{N}$. Let

$$F = \{x_{n_i}: i < k\} \cap \{x_{n_im}: i < k, \ m \geq m_i\}.$$

Then $F$ is compact in $X$, there is a neighborhood $W$ of $p$ in $X$ such that $W \cap F = \emptyset$, so $W \cap (X \setminus \text{cl}(G_k)) \cap T = \emptyset$, hence $T$ is closed in $X$, and $\sigma T$ is homeomorphic to $S_2$. $\square$

Since a closed subspace of a sequential space is sequential, the foregoing proof gives:

**Corollary 2.6.** Let $X$ be a space in which each point is regular $G_6$. If $X$ contains a copy of $S_2$, then $X$ contains a closed copy of $S_2$.

For a space $X$, let $\varphi$ be a family of subsets of $X$. $\varphi$ is a network of $x$ in $X$ if $x \in \bigcap \varphi$ and whenever $G$ is open in $X$ with $x \in G$, then $P \subset G$ for some $P \in \varphi$.

**Definition 2.7.** Let $\varphi = \bigcup \{\varphi_x: x \in X\}$ be a family of subsets of $X$ which satisfies that for each $x \in X$,

1. $\varphi_x$ is a network of $x$ in $X$,
2. if $U, V \in \varphi_x$, then $W \subset U \cap V$ for some $W \in \varphi_x$.

$\varphi$ is a sequentially open network (respectively, a universal cs-network) for $X$ if each element of $\varphi_x$ is a sequentially open subset (respectively, a sequential barrier of $x$) in $X$. A space $X$ is a sof-countable space (respectively, a universally csf-countable space) if $X$ has a sequentially open network (respectively, universal cs-network) $\varphi$ such that each $\varphi_x$ is countable.
Obviously, a space is a first-countable space if and only if it is a sof-countable and sequential space. $S_2$ is not sof-countable. The following two corollaries follow easily from Lemma 2.3, Theorems 2.4 and 2.5.

**Corollary 2.8.** The following are equivalent for a space $X$:

1. $\sigma X$ is a first-countable space.
2. $X$ is a sof-countable space.
3. $X$ is a universally csf-countable space and contains no subspace having $S_2$ as its sequential coreflection.

**Corollary 2.9.** A (regular) space $X$ has a $\sigma$-locally finite sequentially open network if and only if $X$ has a $\sigma$-locally finite universal cs-network and contains no (closed) subspace having $S_2$ as its sequential coreflection.

**Remark 2.10.** If a space $X$ has a $\sigma$-locally finite sequentially open network, then $\sigma X$ has a $\sigma$-locally finite space. But its inverse proposition is not hold. For example, $\sigma(\beta N)$ is a discrete space, and $\beta N$ is not a $\sigma$-space.

**Definition 2.11.** Let $X$ be a space, and let $\mathcal{G}$ be a cover of $X$. $\mathcal{G}$ is a $k$-network for $X$ if, whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \bigcup \mathcal{G} \subset U$ for some finite $\mathcal{G} \subset \mathcal{G}$.

**Theorem 2.12.** Suppose $X$ has a point-countable $k$-network. If $\sigma X$ contains no closed copy of $S_2$, then $\sigma X$ is a Fréchet space.

**Proof.** Suppose $\mathcal{G}$ is a point-countable $k$-network for $X$. If $\sigma X$ is not a Fréchet space, by Theorem 2.4, $X$ contains a subspace $C$ having $S_2$ as its sequential coreflection. Put

$$
C = \{x\} \cup \{x_n: n \in \mathbb{N}\} \cup \{x_{nm}: n, m \in \mathbb{N}\},
$$

$$
K = \{x\} \cup \{x_n: n \in \mathbb{N}\},
$$

$$
\mathcal{A} = \{P \in \mathcal{G}: P \cap \{x_{nm}: n, m \in \mathbb{N}\} \neq \emptyset \text{ and } \overline{P} \cap K = \emptyset\}.
$$

The $\mathcal{A}$ is countable. Let $\mathcal{A} = \{P_k: k \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that $\{x_{nm}: m \geq m_n\} \subset X \setminus \bigcup_{k \leq n} \overline{P}_k$. Take

$$
S = K \cup \{x_{nm}: m \geq m_n\}.
$$

Then $\sigma S$ is homeomorphic to $S_2$. If $\sigma S$ is not closed in $\sigma X$, there is a sequence $\{x_{n_i, m_i}\}$ in $S$ with $x_{n_i, m_i} \to x' \notin S$. We may assume that $n_{i+1} > n_i$. Put

$$
K_1 = \{x'\} \cup \{x_{n_i, m_i}: i \in \mathbb{N}\}.
$$

Then $K_1 \setminus K = \emptyset$, there is an open set $U$ in $X$ with $K_1 \subset U \subset \overline{U} \subset X \setminus K$, thus $K_1 \subset \bigcup \mathcal{G}' \subset U$ for some finite $\mathcal{G}' \subset \mathcal{G}$, so $P \cap K_1$ is infinite for some $P \in \mathcal{G}'$, hence $P = P_j$ for some $j \in \mathbb{N}$, and $x_{n_i, m_i} \notin P$ for each $n_i > j$, a contradiction. Therefore $\sigma S$ is closed in $\sigma X$. \[\square\]
**Corollary 2.13.** Suppose $X$ is a $k$-space with a point-countable $k$-network, then $X$ is a Fréchet space if and only if $X$ contains no closed copy of $S_2$.

**Proof.** Every $k$-space with a point-countable $k$-network is a sequential space [8, Corollary 3.4]. □

**Example 2.14.** There exist a compact, sequential space $X$ and its subspace $M$ such that

1. $X$ contains no copy of $S_2$ or $S(\omega)$.
2. $\sigma M$ is homeomorphic to $S_2$.
3. $M$ has a countable universal cs-network.

**Proof.** By Example 7.1 in [7], let $\psi(\mathcal{N}) = \mathcal{N} \cup A$ be the Isbell’s space, and let $X = \psi(\mathcal{N}) \cup \{a\}$ be the one-point compactification of $\psi(\mathcal{N})$, then $X$ is a compact, sequential space. $X$ contains no copy of $S_2$ or $S(\omega)$ by Corollary 3.10 in [16]. Take an infinite subset $\{A_n: n \in \mathcal{N}\} \subset A$, then the $\{A_n\}$ converges to $a$ in $X$ because $A$ is closed discrete in $\psi(\mathcal{N})$. For each $n \in \mathcal{N}$, put

$$A_n = \{a_{nm}: m \in \mathcal{N}\}.$$

Then the $\{a_{nm}\}$ converges to $A_n$ in $\psi(\mathcal{N})$. Let

$$M = \{a\} \cup \{A_n: n \in \mathcal{N}\} \cup \{a_{nm}: n, m \in \mathcal{N}\}.$$ 

Since any subsequence of $\{a_{nm}\}$ does not converge to $a$ in $X$, by Theorem 2.4, $\sigma M$ is homeomorphic to $S_2$. For each $x \in M$, let

$$\varphi_x = \begin{cases} 
\{\{a\} \cup \{A_n: n \geq i\}: i \in \mathcal{N}\}, & x = a \\
\{\{A_n\} \cup \{a_{nm}: m \geq i\}: i \in \mathcal{N}\}, & x = A_n, \ n \in \mathcal{N} \\
\{\{a_{nm}\}\}, & x = a_{nm}, \ n, m \in \mathcal{N}.
\end{cases}$$

Then $\bigcup \{\varphi_x: x \in X\}$ is a countable universal cs-network for $M$.

$M$ is not sof-countable by Corollary 2.8. $\text{cl}_s(\mathcal{N})$ is not a sequentially closed subset of $X$ because $\text{cl}_s(\mathcal{N}) = \psi(\mathcal{N})$. □

3. On the sequential fan $S(\omega)$

**Definition 3.1.** Call a subspace of a space a fan (at a point $x$) if it consists of a point $x$, and a countably infinite family of disjoint sequences converging to $x$. Call a subset of a fan a diagonal if it is a convergent meeting infinitely many of the sequences converging to $x$ and converges to some point in the fan.

A fan at a point $x$ in a space $X$ is called a countable sheaf at $x$ in [3,4]. If $X$ is a fan, then each point of $X$ is regular $G_\delta$. $S(\omega)$ is a fan without a diagonal.

**Lemma 3.2.** For a space $X$, $\sigma X$ is homeomorphic to $S(\omega)$ if and only if $X$ is a fan without a diagonal.
Proof. Suppose $\sigma X$ is homeomorphic to $S(\omega)$. Since $S(\omega)$ is a fan without a diagonal, $X$ is a fan without a diagonal. Conversely, suppose $X$ is a fan without a diagonal. Since $\sigma X$ is sequential, it is homeomorphic to $S(\omega)$. □

Lemma 3.3. Suppose $X$ contains a fan $S$ at a point $x$ without a diagonal converging to $x$. If $x$ is regular $G_\delta$ in $X$, then $S$ contains a closed subspace $T$ of $X$ such that $\sigma T$ is homeomorphic to $S(\omega)$.

Proof. Let $S = \{x\} \cup \{x_{nm}: n, m \in \mathcal{N}\}$, where the sequence $\{x_{nm}\}$ converges to $x$ for each $n \in \mathcal{N}$. There is a sequence $\{W_n\}$ of open neighborhoods of $x$ in $X$ with $\{x\} = \bigcap\{\text{cl}(W_n): n \in \mathcal{N}\}$. For each $n \in \mathcal{N}$, there is $m(1, n) \in \mathcal{N}$ with $x_{nm(1,n)} \in W_{n+1}$.

Denote $D_1 = \{x_{nm(1,n)}: n \in \mathcal{N}\}$, and $V_1 = X \setminus D_1$, then any subsequence of $D_1$ does not converge to $x$, thus $V_1$ is a sequential barrier of $x$ in $X$. By inductive method, we can construct $D_i = \{x_{nm(1,n)}: n \in \mathcal{N}\}$, and $V_i = X \setminus (D_1 \cup D_2 \cup \cdots \cup D_i)$ such that $x_{nm(i+1,n)} \in W_{n+i+1} \cap V_i$ and $m(i,n) < m(i+1,n)$ for each $i \in \mathcal{N}$. Then the sequence $\{x_{nm(i,n)}: i \in \mathcal{N}\}$ converges to $x$ for each $n \in \mathcal{N}$, and $x_{nm(i,n)} \in W_k$ if $n+i \geq k$. Let

$$T = \{x\} \cup \{x_{nm(i,n)}: i, n \in \mathcal{N}\}.$$ 

Then $T \setminus W_k$ is finite for each $k \in \mathcal{N}$, thus $p \in \text{cl}(W_k)$ when $p$ is an accumulation point of $T$ in $X$, so $p = x$, i.e., $x$ is a unique accumulation point of $T$ in $X$. Therefore, $T$ is closed in $X$, and $\sigma T$ is homeomorphic to $S(\omega)$. □

Corollary 3.4. Let $X$ be a space in which each point is regular $G_\delta$. If $X$ contains a copy of $S(\omega)$, then $X$ contains a closed copy of $S(\omega)$.

Definition 3.5.

1. A space $X$ is an $\alpha_1$-space [3,4] if $T - \{x\} \cup (\bigcup \{T_n: n \in \mathcal{N}\})$ is a fan at $x$ of $X$, where each sequence $T_n$ converges to $x$, then there exists a sequence $S$ converging to $x$ such that $T_n \setminus S$ is finite for each $n \in \mathcal{N}$.

2. A space $X$ is an $\alpha_4$-space [3,4] if every fan at $x$ of $X$ has a diagonal converging to $x$.

3. A space $X$ is a countably bisequential space (or a strong Fréchet space) [13] if, whenever $\{A_n\}$ is a decreasing sequence of subsets of $X$ and $x \in \bigcap\{\text{cl}(A_n): n \in \mathcal{N}\}$, there is a sequence $\{x_n\}$ converging to $x$ with $x_n \in A_n$ for each $n \in \mathcal{N}$.

Clearly, each $\alpha_1$-space is an $\alpha_4$-space, and a space is countably bisequential if and only if it is a Fréchet and $\alpha_4$-space. $X$ is an $\alpha_4$-space if and only if $\sigma X$ is an $\alpha_4$-space.

By Lemma 3.3, we have that

Theorem 3.6. The following are equivalent for a space $X$ (in which each point is regular $G_\delta$):

1. $X$ is an $\alpha_4$-space.
2. Every fan of $X$ has a diagonal.
3. $X$ contains no (closed) subspace having $S(\omega)$ as its sequential coreflection.
Corollary 3.7. Let $X$ be a space (in which each point is regular $G_\delta$). $\sigma X$ is countably bisequential if and only if $X$ contains no (closed) subspace having $S_2$ or $S(\omega)$ as its sequential coreflection.

Theorem 3.8. Suppose $X$ has a point-countable $k$-network. If $\sigma X$ contains no closed copy of $S(\omega)$, then $X$ is an $\alpha_4$-space.

Proof. Suppose $\varphi$ is a point-countable $k$-network for $X$. If $X$ is not an $\alpha_4$-space, by Definition 3.5, there is a fan at $x$ of $X$ without a diagonal converging to $x$. Put

$$S = \{x\} \cup \{x_{nm}: n, m \in \mathbb{N}\},$$
$$\mathcal{R} = \{P \in \varphi: P \cap \{x_{nm}: n, m \in \mathbb{N}\} \neq \emptyset \text{ and } x \notin P\} = \{P_k: k \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that $\{x_{nm}: m \geq m_n\} \subset X \setminus \bigcup_{k \leq n} P_k$. Take

$$T = \{x\} \cup \{x_{nm}: m \geq m_n\}.$$

Then $T$ is a fan at $x$ of $X$ without a diagonal converging to $x$. If there is a sequence $\{x_{n_im_i}\}$ in $T$ with $x_{n_{i+1}m_{i+1}} \to x' \neq x$. We may assume that $n_{i+1} > n_i$. So there exists $P \in \mathcal{R}$ such that $P \cap \{x_{n_im_i}: i \in \mathbb{N}\}$ is infinite, a contradiction. Hence $\sigma T$ is a closed subspace of $\sigma X$, and is homeomorphic to $S(\omega)$. □

Corollary 3.9. Suppose $X$ is a $k$-space with a point-countable $k$-network.

1. $X$ is an $\alpha_4$-space if and only if $X$ contains no closed copy of $S(\omega)$.

2. $X$ is a first-countable space if and only if $X$ contains no closed copy of $S_2$ and $S(\omega)$.

3. $X$ is a first-countable space if and only if $X^\omega$ is a $k$-space.

Proof. Since every $k$-space with a point-countable $k$-network is sequential [8, Corollary 3.4], (1) holds by Theorem 3.8.

If $X$ contains no closed copy of $S_2$ and $S(\omega)$, by (1) and Corollary 2.13, $X$ is countably bisequential. For each $p \in X$, declaring every point $x \in X$, $x \neq p$ isolated and $p$ having old neighborhoods we get a regular countably bisequential topology $\tau$ on $X$ and $X$ has a point-countable $k$-network in this topology. By Corollary 3.6 in [8], $X$ is first-countable at $p$ in the topology $\tau$ and thus in its original topology, (2) holds.

If $X^\omega$ is a $k$-space, $X$ contains no closed copy of $S_2$ and $S(\omega)$ by Proposition 4.2 in [19], hence $X$ is first-countable and (3) holds. □

Corollary 3.9(2) answers a question in [12]. By Corollary 3.9(2), Theorem 6.1 in [8] and Theorem 9.8 in [13], we have the following corollary, which improves some theorems in [20].

Corollary 3.10. Suppose $X$ is a quotient $s$-image of a metric space. $X$ has a point-countable base if and only if $X$ contains no (closed) copy of $S_2$ and $S(\omega)$.
Definition 3.11. Let \( \wp = \bigcup \{ \wp_x : x \in X \} \) be a family of subsets of \( X \) which satisfies the conditions (1) and (2) in Definition 2.7. \( \wp \) is a weak base \cite{2} for \( X \) if a necessary and sufficient condition for \( G \subset X \) to be open in \( X \) is that, for each \( x \in G \), \( P \subset G \) for some \( P \in \wp_x \). \( \wp \) is a cs-network for \( X \) if, given an open neighborhood \( G \) of \( x \) and a sequence \( \{ x_n \} \) converging to \( x \), there are \( P \in \wp_x \) and \( n \in \mathbb{N} \) such that \( x_n \in P \subset G \) for all \( n \geq i \). A space is a gf-countable space \cite{2} (respectively, a csf-countable space) if \( X \) has a weak base (respectively, a cs-network) \( \wp \) such that each \( \wp_x \) is countable. A space is a g-metrizable space \cite{8} (respectively, an \( \aleph \)-space \cite{4}) if it is a regular space having a \( \sigma \)-locally finite weak base (respectively, cs-network).

Every g-metrizable space is gf-countable. Every \( \aleph \)-space is csf-countable. The following lemma can be checked directly.

Lemma 3.12. Let \( \wp \) be a cover of a space \( X \). If \( \wp \) is a weak base for \( X \), then \( \wp \) is a universal cs-network for \( X \). If \( X \) is a sequential space and \( \wp \) is a universal cs-network for \( X \), then \( \wp \) is a weak base.

Theorem 3.13. The following are equivalent for a space \( X \):

1. \( \sigma X \) is a gf-countable space.
2. \( X \) is a universally csf-countable space.
3. \( X \) is a csf-countable and \( \alpha_1 \)-space.
4. \( X \) is a csf-countable and \( \alpha_4 \)-space.

Proof. (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (4) are obvious.

(2) \( \Rightarrow \) (3) Suppose \( X \) is a universally csf-countable space. Let \( F = \{ x \} \cup (\bigcup \{ \tau_n : n \in \mathbb{N} \} \) be a fan at \( x \) of \( X \), where each sequence \( T_n \) converges to \( x \). Let \( \{ P_n : n \in \mathbb{N} \} \) be a decreasing universal cs-network at \( x \) in \( X \), and \( S_n = T_n \cap P_n \) for each \( n \in \mathbb{N} \). \( S = \bigcap_{n \in \mathbb{N}} S_n \) is a sequence converging to \( x \), and \( T_n \setminus S \) is finite for each \( n \in \mathbb{N} \). Hence \( X \) is an \( \alpha_1 \)-space.

(4) \( \Rightarrow \) (1) Suppose \( X \) is a csf-countable and \( \alpha_4 \)-space. For each \( x \in X \), let \( \wp_x \) be a countable cs-network at \( x \) in \( X \). Put

\[
\mathcal{R}_x = \left\{ \bigcup \wp_x' : \wp_x' \text{ is a finite subset of } \wp_x \text{ and } \wp_x' \text{ is a sequential barrier of } x \text{ in } X \right\}.
\]

If \( \mathcal{R}_x \) is not a network of \( x \) in \( X \), then there exists an open subset \( G \) in \( X \) such that \( x \subset G \) and \( F \not\subset G \) for each \( F \in \mathcal{R}_x \). Denote

\[
\{ P \in \wp_x : P \subset G \} = \{ P_i : i \in \mathbb{N} \}, \quad F_n = \bigcup \{ P_i : i \leq n \}, \quad n \in \mathbb{N}.
\]

Then \( F_n \) is not a sequential barrier of \( x \) in \( X \). Since \( \wp_x \) is a cs-network at \( x \) in \( X \), there are a sequence \( T_i \) converging to \( x \) and \( n_i \in \mathbb{N} \) such that \( T_i \subset P_{n_i+1} \setminus F_{n_i} \), and \( n_{i+1} > n_i \) for each \( i \in \mathbb{N} \). Put

\[
T = \{ x \} \cup \left( \bigcup \{ T_i : i \in \mathbb{N} \} \right).
\]
Then $T$ is a fan at $x$ in $X$. Since $X$ is an $\alpha_4$-space, $T$ has a diagonal \( \{x_k\} \) converging to $x$, there are $i$ and $m \in \mathbb{N}$ such that $x_k \in P_i$ for all $k \geq m$. Take some $k \geq m$ and some $j \geq i$ with $x_k \in T_j$, then $x_k \in P_i \cap (X \setminus F_{n_j}) = \emptyset$, a contradiction. So $\mathcal{R}_x$ is a countable universal cs-network at $x$ in $X$, and $\mathcal{R}_x$ is a countable universal cs-network at $x$ in $\sigma X$. Since $\sigma X$ is sequential, $\sigma X$ is $gf$-countable. 

By Corollary 3.9, Theorem 3.13 and Lemma 7(3) in [10], we have the following corollary which answers a question in [21].

**Corollary 3.14.** Suppose $X$ is a sequential space with a point-countable cs-network. $X$ has a point-countable weak base if and only if $X$ contains no (closed) copy of $S(\omega)$.

**Theorem 3.15.** The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally finite universal cs-network.
2. $X$ is an $\aleph$ and $\alpha_4$-space.
3. $X$ is an $\aleph$ and $\alpha_4$-space.
4. $X$ is an $\aleph$-space and contains no (closed) subspace having $S(\omega)$ as its sequential coreflection.

**Proof.** (1) implies (2) because of Theorem 3.13. (2) implies (3) by Definition 3.5. (3) is equivalent to (4) by Theorem 3.6. We show that (3) $\Rightarrow$ (1). Suppose $X$ is an $\aleph$ and $\alpha_4$-space. Let $\varphi$ be a $\sigma$-locally finite cs-network for $X$ which is closed under finite intersections. By Theorem 3.13, $X$ is universally csf-countable. For each $x \in X$, let $\{Q_n(x): n \in \mathbb{N}\}$ be a universal cs-network at $x$ in $X$. Let

$$\varphi_x = \{P \in \varphi: Q_n(x) \subseteq P \text{ for some } n \in \mathbb{N}\}.$$ 

Then $\varphi_x$ is a universal cs-network at $x$ in $X$ by the proof of Lemma 7(3) in [10], thus $\bigcup \{\varphi_x: x \in X\}$ is a $\sigma$-locally finite universal cs-network for $X$. 

Since $k$-spaces are equivalent to sequential spaces in which each point is $G_\delta$ [13], we have that

**Corollary 3.16** [11]. The following are equivalent for a $k$-space $X$:

1. $X$ is a $g$-metrizable space.
2. $X$ is an $\aleph$ and $\alpha_1$-space.
3. $X$ is an $\aleph$ and $\alpha_4$-space.
4. $X$ is an $\aleph$-space and contains no (closed) copy of $S(\omega)$.

**Corollary 3.17.** A space is a metrizable space if and only if it is a countably bisequential $\aleph$-space.

**Theorem 3.18.** The following are equivalent for a space $X$:

1. $X$ has a countable universal cs-network.
2. $X$ is an $\alpha_1$-space with a countable cs-network.
(3) $X$ is an $\alpha_4$-space with a countable cs-network.

(4) $X$ has a countable cs-network and contains no subspace having $S(\omega)$ as its sequential coreflection.

**Proof.** By Theorem 3.13, Definition 3.5 and Theorem 3.6, we only need to show that (4) implies (1). Let $\wp$ be a countable cs-network for $X$ which is closed under finite unions. For each $x \in X$, put

$$\wp_x = \{ P \in \wp : P \text{ is a sequential barrier at } x \text{ in } X \}.$$ 

If $\wp_x$ is not a network of $x$ in $X$, by the proof in Theorem 3.13, we has a fan $T$ at $x$ in $X$. Using the same notation in the proof in Theorem 3.13, if $D$ is a diagonal of $T$ converging to $d$, then $\{x, d\} \cup D \subset P \subset G$ for some $P \in \wp$, thus $P = P_i$ for some $i \in N$. Take some $j \geq i$ and $d' \in D \cap T_j$, then $d' \in P_i \cap T_j \subset P_i \cap (X \setminus F_{n_j}) = \emptyset$, a contradiction. This show that $T$ has not a diagonal. By Lemma 3.2, $\sigma T$ is homeomorphic to $S(\omega)$, a contradiction. Hence $\wp_x$ is a network of $x$ in $X$, and $X$ has a countable universal cs-network. 

**Example 3.19.** There are a compact, sequential space $Y$ and its subspace $T$ such that

1. $Y$ contains no copy of $S_2$ or $S(\omega)$.
2. $\sigma T$ is homeomorphic to $S(\omega)$.
3. $T$ has a countable cs-network.

**Proof.** By the same notation in Example 2.14, let $A = \{A_n : n \in N\}$. Take $Y = X/A$ and let $f : X \to Y$ be the natural quotient map, then $Y$ is a compact, sequential space, and $Y$ contains no copy of $S_2$ or $S(\omega)$ by Corollary 3.10 in [16]. Let $T = f(M)$, then $T$ has a countable cs-network and $\sigma T$ is homeomorphic to $S(\omega)$. 

**Acknowledgement**

This paper is a revision of the original paper "$\aleph$-spaces and copy of $S_\omega$ or $S_2$" by the author, main results of which are Theorems 1.3 and 1.4 in this paper. The author would like to thank the referee for proposing some recommendations for rewriting the original paper, which contain the ideas of comb and fan, and some weakly first-countable properties on the sequential coreflection of a space, and so forth.

**References**